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The Realineituhedron

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Abstract: Perturbative scattering amplitudes in gauge theories have remarkable simplicity and hidden infinite dimensional symmetries that are completely obscured in the conventional formulation of field theory using Feynman diagrams. This suggests the existence of a new understanding based on a simple geometrical object. In this note we provide such an understanding in the linear limit, which we identify as "the volume" of a new mathematical object–the Realineituhedron.

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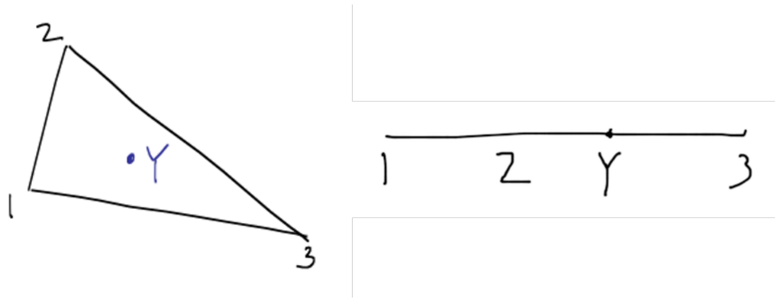
1. Introduction

Scattering amplitudes in gauge theories are amongst the most fundamental observables in physics. The textbook approach to computing these amplitudes in perturbation theory, uses Feynman diagrams, which initiated a huge industry in child-like drawings by professional physicists. Over the last quarter-century it has become clear that making the drawings in two-dimensions is a defect of the Feynman diagram approach to this physics, and is not present in the final amplitudes themselves, which are astonishingly simpler than indicated from the diagrammatic expansion [1–7]. In fact, the physically observable quantity must be a real number, a feature foreshadoewed by the Hermitian postulate of quantum mechanics. Is it possible that there is some geometrical object is able to capture the Hermitian nature of these operators–indeed, is it able to represent all fundamental observables?

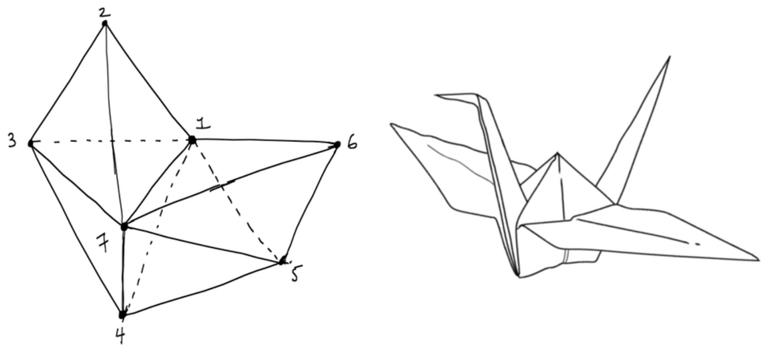
In this note we finally realize this picture. We will introduce a new mathematical object whose "volume" directly computes the value of any fundamental observable. We call this object the "Realineituhedron", to denote its connection both to reality and elementary geometry.

2. Triangles → The Real Line $\mathbb{R}$

To begin with, let us start with the simplest and most familiar geometric object of all, a triangle in two dimensions. Thinking projectively, the vertices would all be squashed onto a line.



One obvious generalization of the triangle is to an $(n-1)$ dimensional simplex in a general projective space. We can also talk about the very closely related space of positive matrices $M_+(k,n)$, which are just $k \times n$ matrices with all positive ordered minors. The only difference with the positive Grassmannian is that we actually know what matrices are. An $(n-1)$ dimensional simplex can often give rise to a beautiful object, though one must take care that the dashed lines represent a valley fold, while solid lines represent a mountain fold.



3. Why not Negativity?

We have motivated the structure of the realineituhedron by mimicking the geometric idea of the "inside" of a convex polygon squashed onto a line. However, there is a deeper origin related to the curious sum of integers.

$$\sum_{ii=0}^{\infty}(1\,ii) = -\frac{1}{12}$$

What!? That's crazy, how can a sum over positive numbers be negative? If positivity is such a good guiding principle, why not negativity?

4. “Volume” as a Canonical Distance

Before discussing how to determine the value of an observable from the geometry, let us define the notion of a “volume” associated with the realineituhedron. As should by now be expected, we will merely generalize a simple existing idea from the world of triangles and polygons squashed onto a line. The usual notion of “area” is obviously not projectively meaningful. However there is a closely related idea that is. For the squashed triangle, we can consider a rational 1-form in Y-space, which measures the distance between two points on the boundary. We denote this “volume” Ω.

$$\Omega = \int_a^b dY$$

5. The Realamplitude

Now that we have defined central objects in our story: the realineituhedron, together with the associated form Ω that is loosely speaking its “volume”. The value of any physical observable can can be directly extracted from Ω. To see how, note that we we can always use a transformation $x \to x' = \lambda x$, where $x' = 1$ and the value of x is simply $1/\lambda$.

It is trivial to imagine what we might mean by hiding a single particle, but as we will see momentarily, the idea of hiding particles is only natural if we hide pairs of adjacent particles inside of some ascii art. Can you find the hidden particles?

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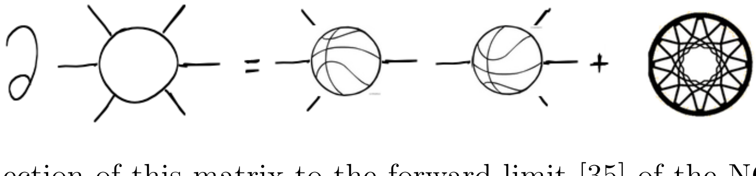
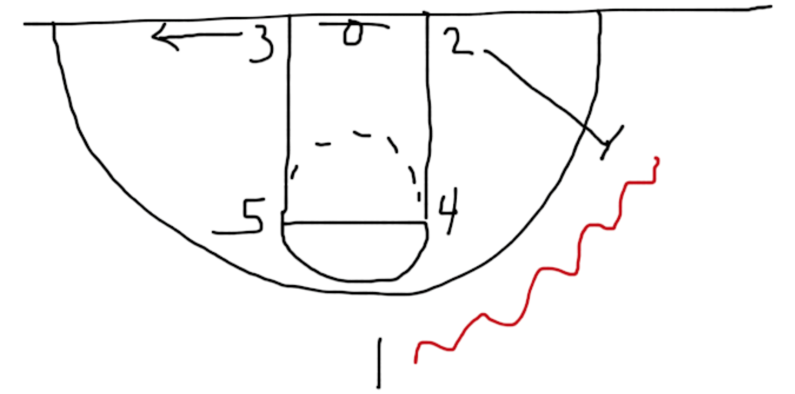
6. Locality and Unitarity from Linearity

Locality and unitarity are encoded in the linear geometry of the realineituhedron in a beautiful way. As is well-known, locality and unitarity are directly reflected in the singularity structure of the integrand for scattering amplitudes. Unitarity is reflected in what happens as poles are approached, schematically we have

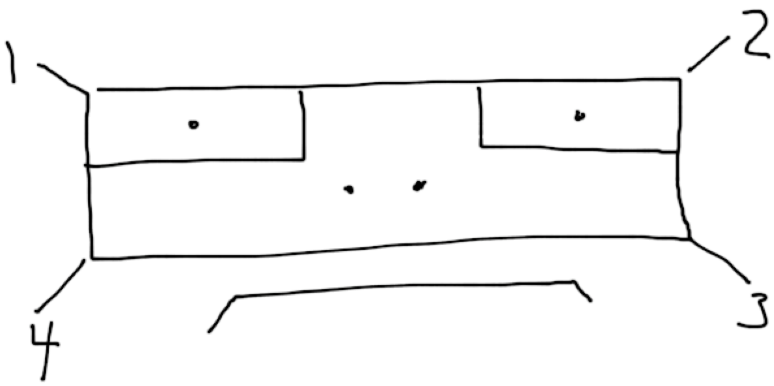


The connection of this matrix to the forward limit [35] of the NCAA bra-ket is simple. Surprisingly, this leads to a set of recursion relations or “plays” that can be utilized within the NCAA setting.



7. The Master Realineituhedron $\mathbb{R}^2$

If we consider general even m, we can also generalize the notion of "hiding particles" in an obvious way: adjacent particles can be hidden in even numbers. This leads us to a bigger space in which to embed the generalized loop realineituhedron, schematically:



We can call this space $\mathbb{R}^2$. The m = 1 realineituhedron is again just a particular face of this object. It would be interesting to see whether this larger space has any interesting role to play in understanding the m = 1 geometry relevant to physics.

Acknowledgements

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